Prestacks.

We start by taking as a definition of affine schemes the opposite of the ∞-category of commutative rings.

$$\mathrm{Sch}^{\mathrm{aff}} := (\mathrm{CAlg})^{\mathrm{op}}$$

, we will drop k from the notation, but we fixed a base field.

As we already discussed, for each $n \geq 0$ one has fully faithful functors:

$$\mathrm{CAlg}(\mathrm{Vect}^{\geq n, \leq 0}) \hookrightarrow \mathrm{CAlg}$$

we let

!!

$$^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}$$

denote the corresponding functor and

$$\tau^{\leq n} : \mathrm{Sch}^{\mathrm{aff}} \to {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}} \to \mathrm{Sch}^{\mathrm{aff}}$$

the composite.

In particular, $^{cl}\mathrm{Sch}^{\mathrm{aff}} = {}^{\leq 0}\mathrm{Sch}^{\mathrm{aff}}$ from the fact that discrete commutative algebras were equivalent to ordinary commutative algebras.

<u>Def'n:</u> A <u>prestack</u> $\mathcal{X}$ is a functor from the opposite of the category of affine schemes to spaces, i.e. $\mathcal{X} : \mathrm{Sch}^{\mathrm{aff}\,\mathrm{op}} \to \mathrm{Spc}$.

Let $\mathrm{PStk} := \mathrm{Fun}(\mathrm{Sch}^{\mathrm{aff}\,\mathrm{op}}, \mathrm{Spc})$ denote the ∞-category of such.

<u>Rk:</u> The above definition is so general that it is essentially impossible to say something about an arbitrary prestack. It is useful though in the sense that one should sometimes strive to make sense of certain constructions in this generality so that they apply to "dire" objects that one might be confronted with.

__Example:__ For X a ~~smooth~~ ~~proper~~ scheme non-empty let for each finite set I let consider X^I the product of $|I|$ copies of X. For a map $\alpha: I \twoheadrightarrow J$ one has a map:

$$\delta: X^J \to X^I \quad \text{given by}$$
$$(x_j)_J \mapsto (\delta((x_i)_J)_i)_I . \qquad \delta(x_j)_i = x_j \quad \forall i \in \alpha^{-1}(j).$$

$$\mathrm{Ran}(X) := \operatorname*{colim}_{I \in \mathrm{Fin}^{surj}} X^I .$$

$\mathrm{Ran}(X)$ normally is nothing but a prestack, i.e. is not an ind-scheme, does not satisfy descent, so not a stack (see below). It is however a useful object in the geometric Langlands program. (See also Gaitsgory-Lurie work on Tamagawa numbers.)

We will discuss some conditions that make prestacks more reasonable.

Let ${}^{\leq n}(-) : \mathrm{Fun}(\mathrm{Sch}^{aff, op}, \mathrm{Spc}) \to \mathrm{Fun}(({}^{\leq n}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc})$ denote the restriction via ${}^{\leq n}\mathrm{Sch}^{aff} \hookrightarrow \mathrm{Sch}^{aff}$.

$$\mathrm{Fun}(({}^{\leq n}\mathrm{Sch}^{aff})^{op}, \mathrm{Spc}) \,\|\, {}^{\leq n}\mathrm{PStk}$$

One has a <u>fully faithful left adjoint</u> $\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{aff} \hookrightarrow \mathrm{Sch}^{aff}} : {}^{\leq n}\mathrm{PStk} \to \mathrm{PStk}$.
(b/c. $\mathrm{id}_{{}^{\leq n}\mathrm{PStk}} \to {}^{\leq n}(-) \circ \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{aff} \hookrightarrow \mathrm{Sch}^{aff}}$ is an isom.).

Similarly, one denotes by $\tau^{\leq n} := \mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{aff} \hookrightarrow \mathrm{Sch}^{aff}} \circ {}^{\leq n}(-)$.

Informally, one has

$$\mathrm{LKE}_{{}^{\leq n}\mathrm{Sch}^{aff} \hookrightarrow \mathrm{Sch}^{aff}}(\mathcal{X})(S) = \operatorname*{colim}_{\{S \to S' \mid S' \in {}^{\leq n}\mathrm{Sch}^{aff}\}} \mathcal{X}(S').$$

__Def'n:__ One says $\mathcal{X} \in \mathrm{PStk}$ is <u>n-coconnective</u> if

$$\tau^{\leq n}(\mathcal{X}) \xrightarrow{\simeq} \mathcal{X} \qquad \text{is an isomorphism, i.e.}$$

one can recover the data of $\mathcal{X}$ from its value on n-coconnective affine schemes. (${}^{\leq n}\mathrm{Sch}^{aff}$).

Analogous to the subcategories of affine schemes, we say $\mathcal{X}$ is a <u>classical prestack</u>

if $\mathcal{X}$ is 0-coconnective. We also denote ${}^{\leq 0}\mathrm{PStk} =: {}^{cl}\mathrm{PStk}$. , ~~and~~

The composite:

$$\mathrm{PStk} \xrightarrow{{}^{cl}(-)} {}^{cl}\mathrm{PStk} \to \mathrm{PStk} \quad \text{by} \quad \tau^{cl}(-), \text{ and}$$

The restriction $\mathrm{PStk} \xrightarrow{{}^{\leq 0}(-)} {}^{cl}\mathrm{PStk}$ by ${}^{cl}(-)$.

<u>Example:</u> Any $S \in {}^{\leq n}\mathrm{Sch}^{aff}$ is an n-coconnective prestack via the Yoneda embedding, i.e. $\mathrm{Hom}_{\mathrm{Sch}^{aff}}(-, S) : \mathrm{Sch}^{aff} \to \mathrm{Spc}$.

The next condition is something that we expect from ~~useful~~ prestacks from a geometric nature. For $S \in \mathrm{Sch}^{aff}$

Consider the functor:
$$\mathbb{Z}_{\geq 0} \to \mathrm{Sch}^{aff}_{/S}$$
$$n \mapsto \tau^{\leq n}(S).$$

<u>Def'n:</u> A prestack $\mathcal{X}$ is <u>convergent</u> if the canonical map:

$$\mathcal{X}(S) \to \lim_{n \geq 0} \mathcal{X}(\tau^{\leq n}(S)) \quad \text{is an isomorphism.}$$

Let ${}^{<\infty}\mathrm{Sch}^{aff} \subset \mathrm{Sch}^{aff}$ be the subcategory of <u>eventually coconnective</u> affine schemes, i.e. $S \in {}^{<\infty}\mathrm{Sch}^{aff}$ if $\tau^{\leq n}(S) = S$ for some n.

<u>Exercise:</u> (i) $\mathcal{X}$ is convergent iff the canonical map

$$\mathcal{X} \to \mathrm{RKE}_{{}^{<\infty}\mathrm{Sch}^{aff} \hookrightarrow \mathrm{Sch}^{aff}} \left(\mathcal{X}|_{{}^{<\infty}\mathrm{Sch}^{aff}} \right) \quad \text{is an equivalence;}$$

(ii) any affine scheme is convergent;

(iii) The prestack: $\mathcal{Q}$ defined as the composite:

$$\mathrm{Sch}^{aff,op} \xrightarrow[\mathrm{QCoh}(-)^{*}]{} \mathrm{Cat}_\infty \xrightarrow[(-)^{\simeq}]{} \mathrm{Spc}. \quad \text{is } \underline{\text{not}} \text{ convergent.}$$

Rk: The inclusion ${}^{conv}\mathrm{PStk} \hookrightarrow \mathrm{PStk}$. has a ~~right~~ left adjoint given by
$${}^{conv}\mathrm{PStk} = \{\mathcal{X} \in \mathrm{PStk} \mid \mathcal{X} \text{ convergent}\}.$$
$${}^{conv}\mathcal{X}(S) \simeq \lim_{n \geq 0} \mathcal{X}(\tau^{\leq n}(S)).$$
[SAG. 17.3.3.2].

Finiteness conditions. We start by discussing these conditions for algebras.

Def'n: Given $S \in Sch^{aff}$, where $S = Spec(A)$ $A \in CAlg$, we say

- S is of finite type if (i) $H^0(A)$ is of finite type $/k$.
  (ii) $H^i(A) = 0$ for $i << 0$.
  (iii) $H^i(A)$ is a f.g. $H^0(A)$-module $\forall i \in \mathbb{Z}$.

let ${}^{<\infty}Sch^{aff}_{ft} \hookrightarrow {}^{<\infty}Sch^{aff} \hookrightarrow Sch^{aff}$ denote the corresponding subcategory.

- S is almost of finite type if (i) $H^0(A)$ is of finite type $/k$;
  (ii) $H^i(A)$ is a f.g. $H^0(A)$-mod. $\forall i \in \mathbb{Z}$.

Equivalently, $\forall n \geq 0$, ${}^{\leq n}S$ is of finite type.

Examples: (i) $k[\varepsilon]$ w/ $|\varepsilon| = -1$ is of finite type.
(ii) $k[y]$ w/ $|y| = -2$ is almost of finite type.

Prop: (Noeth. approx.) For any $S \in {}^{\leq n}Sch^{aff}$ one has $S \xrightarrow{\simeq} \lim_{S_\alpha \to S,\ S_\alpha \in {}^{\leq n}Sch^{aff}_{ft}} S_\alpha$, i.e. $\operatorname{Pro}({}^{\leq n}Sch^{aff}_{ft}) \simeq {}^{\leq n}Sch^{aff}$.

These conditions immediately generalize to prestacks.

Def'n: - For $\mathcal{Y} \in {}^{\leq n}PStk$ for some $n \geq 0$, $\mathcal{Y}$ is said to be locally of finite type if the canonical map:

$$\underset{{}^{\leq n}Sch^{aff}_{ft} \hookrightarrow {}^{\leq n}Sch^{aff}}{LKE} \left(\mathcal{Y}|_{{}^{\leq n}Sch^{aff}_{ft}} \right) \xrightarrow{\simeq} \mathcal{Y}$$ is an isomorphism.

- For $\mathcal{X} \in PStk$, $\mathcal{X}$ is said to be locally almost of finite type if
  (i) $\mathcal{X}$ is convergent
  (ii) $\forall n \geq 0$, ${}^{\leq n}\mathcal{X} \in {}^{\leq n}PStk_{lft}$, i.e. ${}^{\leq n}\mathcal{X}$ is locally of finite type.

Sanity checks:

Prop: (i) $\mathcal{Y} : ({}^{\leq n}Sch^{aff})^{op} \to Spc$ is locally of finite type iff $\mathcal{Y}$ takes (co)filtered limits to colimits.
(ii) $S \in {}^{\leq n}Sch^{aff}$ is of finite type iff $h_S : ({}^{\leq n}Sch^{aff})^{op} \to Spc$, $T \mapsto Hom(T, S)$ is locally of finite type.
(iii) $Sch^{aff}_{aft} \simeq Sch^{aff} \cap PStk_{laft}$.

Prop 2:

$$\begin{tikzcd} {}^{<\infty}Sch^{aff}_{ft} \arrow[r] \arrow[d, hook, "\iota_{ft}"'] & Spc. \\ {}^{<\infty}Sch^{aff} \arrow[d, hook, "{}^{<\infty}\iota"'] & \\ Sch^{aff} & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{X} \in PStk \text{ is laft.} \arrow[d, Leftrightarrow] \\ \mathcal{X} \xrightarrow{\sim} RKE_{{}^{<\infty}\iota}(\mathcal{X}|_{{}^{<\infty}Sch^{aff}}) \\ RKE_{{}^{<\infty}\iota} \circ LKE_{\iota_{ft}}(\mathcal{X}|_{{}^{<\infty}Sch^{aff}_{ft}}) \arrow[u, "is"'] \end{tikzcd}$$

Exercise: Prove these two propositions. (See Chapter 2, §1 GR-I for details.)

Finally, we discuss the notion of truncatedness.

~~Def~~ Let $P_{\le k}: Spc \xrightarrow{\tau_{\le k}(-)} Spc^{\le k} \to Spc$ denote the composite of truncation for spaces w/ the canonical inclusion.

Def'n: A prestack $\mathcal{X} \in {}^{\le n}PStk$ is said to be k-truncated if it factors as follows:

$$\begin{tikzcd} \mathcal{X}: {}^{\le n}Sch^{aff,op} \arrow[r] \arrow[dr] & Spc^{\le k} \arrow[d, hook] \\ & Spc. \end{tikzcd}$$

We will denote the category of such by ${}^{\le n}PStk_{\le k}$.

Example: (i) For any $S \in {}^{\le n}Sch^{aff}$, h_S is n-truncated as a ${}^{\le n}PStk$.

(ii) For any topological space $X \in Spc$, let

$\underline{X}(S) := X$ be the constant prestack in ${}^{\le n}PStk$.

Then $\underline{X}$ is k-truncated $\Leftrightarrow$ X is k-truncated in any).

Rk: ${}^{cl}PStk_{\le 1}$ is the ordinary category of classical prestacks.
${}^{cl}PStk_{\le 0}$ is the ordinary category of presheaves of sets on ord. affine schemes.